q-Hodge filtrations, Habiro cohomology & ku

Ferdinand Wagner

January 8, 2026

q-Hodge filtrations, Habiro cohomology & ku

Ferdinand Wagner

January 8, 2026

Outline

MAX PLANCK INSTITUTE
FOR MATHEMATICS

Outline

Outline

1. What's Habiro cohomology supposed to do?
2. q-Hodge filtrations & Habiro cohomology
3. q-Hodge filtrations from homotopy theory

⚠ **Disclaimer:** We'll focus on "algebraic" Habiro cohomology; Scholze's "analytic" version will only show up in the end.

Outline

⚠ **Disclaimer:** We'll focus on "algebraic" Habiro cohomology; Scholze's "analytic" version will only show up in the end.

Outline

1. What's Habiro cohomology supposed to do?
2. q-Hodge filtrations & Habiro cohomology
3. q-Hodge filtrations from homotopy theory
4. Refined THH/TC^{-} & analytic Habiro cohomology

⚠ **Disclaimer:** We'll focus on "algebraic" Habiro cohomology; Scholze's "analytic" version will only show up in the end.

1. What's Habiro cohomology supposed to do?

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\ldots$)

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\ldots$) by analysing the cohomology of the associated geometric objects.

Cohomology theories

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\ldots$) by analysing the cohomology of the associated geometric objects.

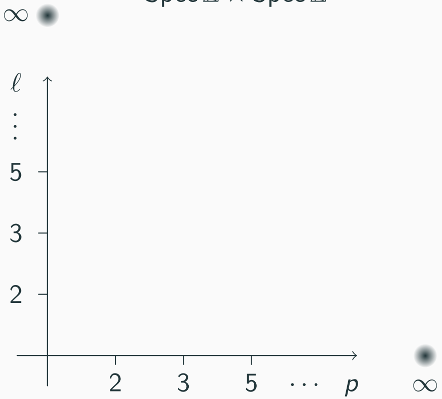
Many cohomology theories:

Cohomology theories

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, ...) by analysing the cohomology of the associated geometric objects.

Many cohomology theories:

“$\overline{\operatorname{Spec}\mathbb{Z}} \times \overline{\operatorname{Spec}\mathbb{Z}}$”



(Scholze)

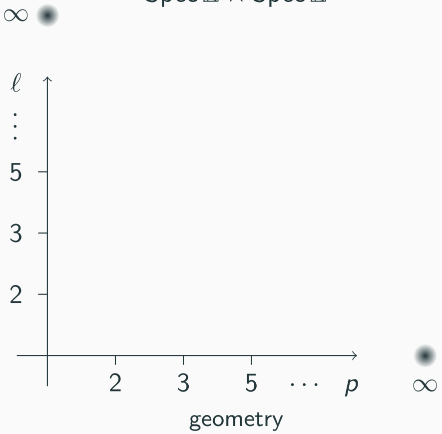
Cohomology theories

MAX PLANCK INSTITUTE
FOR MATHEMATICS

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, ...) by analysing the cohomology of the associated geometric objects.

Many cohomology theories:

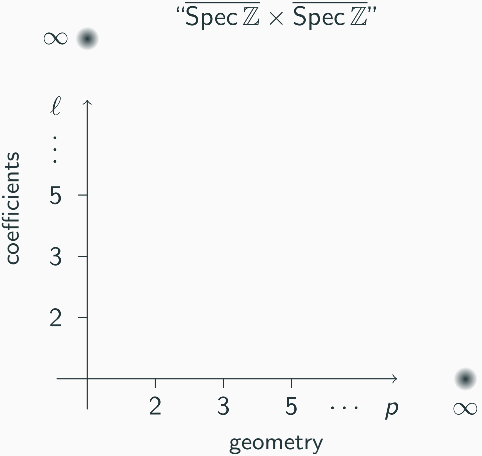
“$\overline{\operatorname{Spec}\mathbb{Z}} \times \overline{\operatorname{Spec}\mathbb{Z}}$”



(Scholze)

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, ...) by analysing the cohomology of the associated geometric objects.

Many cohomology theories:

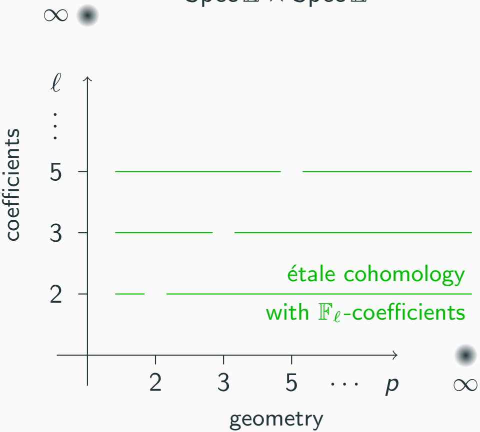


(Scholze)

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, ...) by analysing the cohomology of the associated geometric objects.

Many cohomology theories:

"$\overline{\operatorname{Spec}\mathbb{Z}} \times \overline{\operatorname{Spec}\mathbb{Z}}$"

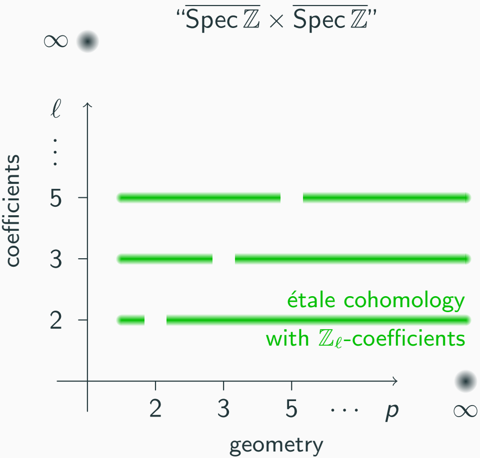


(Scholze)

Cohomology theories

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, …) by analysing the cohomology of the associated geometric objects.

Many cohomology theories:



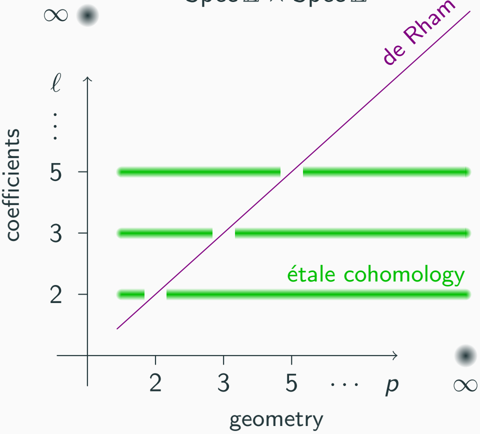
(Scholze)

Cohomology theories

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\ldots$) by analysing the cohomology of the associated geometric objects.

Many cohomology theories:

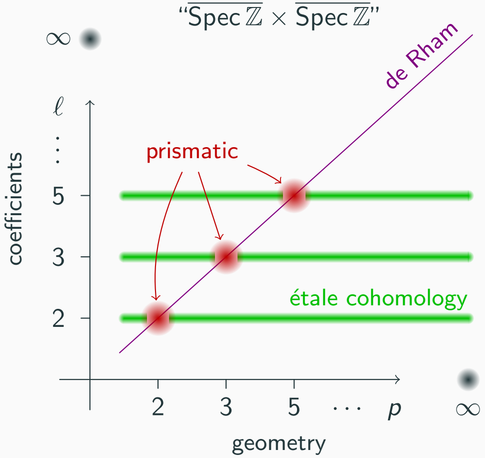
“$\overline{\operatorname{Spec}\mathbb{Z}} \times \overline{\operatorname{Spec}\mathbb{Z}}$”



(Scholze)

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, ...) by analysing the cohomology of the associated geometric objects.

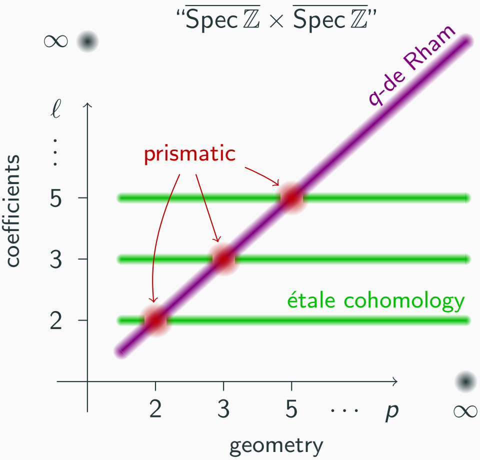
Many cohomology theories:



(Scholze)

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\ldots$) by analysing the cohomology of the associated geometric objects.

Many cohomology theories:



(Scholze)

***q*-de Rham complex** (Bhatt–Scholze)**.** There is a functor

$$q\text{-}\Omega_{(-)}\colon \quad \mathsf{Sm}_{\mathbb{Z}} \quad \longrightarrow \mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big)$$

***q*-de Rham complex** (Bhatt–Scholze)**.** There is a functor

$$q\text{-}\Omega_{(-)}\colon \underset{\substack{\text{smooth}\\ \mathbb{Z}\text{-algebras}}}{\mathsf{Sm}_{\mathbb{Z}}} \longrightarrow \underset{\substack{\text{derived category}\\ \text{of } \mathbb{Z}[\![q-1]\!]}}{\mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big)}$$

q-de Rham complex (Bhatt–Scholze)**.** There is a functor

$$q\text{-}\Omega_{(-)}\colon \underset{\substack{\text{smooth}\\ \mathbb{Z}\text{-algebras}}}{\mathrm{Sm}_{\mathbb{Z}}} \longrightarrow \underset{\substack{\text{derived category}\\ \text{of } \mathbb{Z}[\![q-1]\!]}}{\mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big)}$$

such that:

1. At $q = 1$, $q\text{-}\Omega_R/(q-1) \simeq \Omega^*_R$.

q-de Rham complex (Bhatt–Scholze). There is a functor

$$q\text{-}\Omega_{(-)}\colon \underset{\substack{\text{smooth}\\ \mathbb{Z}\text{-algebras}}}{\mathrm{Sm}_{\mathbb{Z}}} \longrightarrow \underset{\substack{\text{derived category}\\ \text{of } \mathbb{Z}[\![q-1]\!]}}{\mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big)}$$

such that:

1. At $q=1$, $q\text{-}\Omega_R/(q-1) \simeq \Omega_R^*$.
2. For all primes p, $(q\text{-}\Omega_R)_p^\wedge \simeq \mathbb{\Delta}_{\widehat{R}_p[\zeta_p]/(\mathbb{Z}_p[\![q-1]\!],[p]_q)}$.

q-de Rham complex (Bhatt–Scholze)**.** There is a functor

$$q\text{-}\Omega_{(-)}\colon \underset{\substack{\text{smooth}\\ \mathbb{Z}\text{-algebras}}}{\mathrm{Sm}_{\mathbb{Z}}} \longrightarrow \underset{\substack{\text{derived category}\\ \text{of } \mathbb{Z}[\![q-1]\!]}}{\mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big)}$$

such that:

1. At $q=1$, $q\text{-}\Omega_R/(q-1) \simeq \Omega_R^*$.
2. For all primes p, $(q\text{-}\Omega_R)_p^\wedge \simeq \mathbb{\Delta}_{\widehat{R}_p[\zeta_p]/(\mathbb{Z}_p[\![q-1]\!],[p]_q)}$.
3. Rationally, $(q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})^\wedge_{(q-1)} \simeq (\Omega_R^* \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$

q-de Rham complex (Bhatt–Scholze)**.** There is a functor

$$q\text{-}\Omega_{(-)}\colon \underset{\substack{\text{smooth}\\ \mathbb{Z}\text{-algebras}}}{\mathrm{Sm}_{\mathbb{Z}}} \longrightarrow \underset{\substack{\text{derived category}\\ \text{of } \mathbb{Z}[\![q-1]\!]}}{\mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big)}$$

such that:

1. At $q=1$, $q\text{-}\Omega_R/(q-1) \simeq \Omega_R^*$.
2. For all primes p, $(q\text{-}\Omega_R)_p^\wedge \simeq \mathbb{\Delta}_{\widehat{R}_p[\zeta_p]/(\mathbb{Z}_p[\![q-1]\!],[p]_q)}$.
3. Rationally, $(q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})_{(q-1)}^\wedge \simeq (\Omega_R^* \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$,

$$(q\text{-}\Omega_R)_p^\wedge\left[\tfrac{1}{p}\right]_{(q-1)}^\wedge \simeq (\Omega_R^*)_p^\wedge\left[\tfrac{1}{p}\right][\![q-1]\!]\,.$$

q-de Rham complex (Bhatt–Scholze)**.** There is a functor

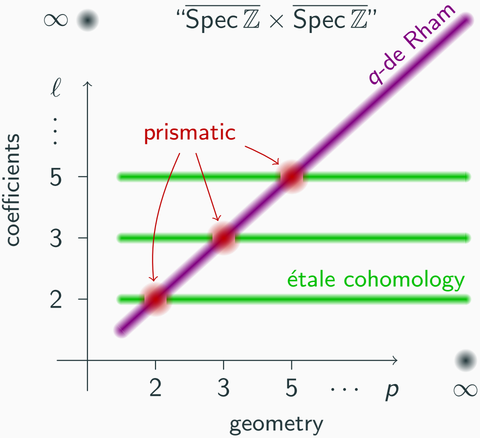
$$q\text{-}\Omega_{(-)}\colon \underset{\substack{\text{smooth}\\ \mathbb{Z}\text{-algebras}}}{\mathrm{Sm}_{\mathbb{Z}}} \longrightarrow \underset{\substack{\text{derived category}\\ \text{of } \mathbb{Z}[\![q-1]\!]}}{\mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big)}$$

such that:

1. At $q=1$, $q\text{-}\Omega_R/(q-1) \simeq \Omega_R^*$.
2. For all primes p, $(q\text{-}\Omega_R)_p^\wedge \simeq \Delta_{\widehat{R}_p[\zeta_p]/(\mathbb{Z}_p[\![q-1]\!],[p]_q)}$.
3. Rationally, $(q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})_{(q-1)}^\wedge \simeq (\Omega_R^* \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$,
$$(q\text{-}\Omega_R)_p^\wedge\left[\tfrac{1}{p}\right]_{(q-1)}^\wedge \simeq (\Omega_R^*)_p^\wedge\left[\tfrac{1}{p}\right][\![q-1]\!]\,.$$

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, ...) by analysing the cohomology of the associated geometric objects.

Many cohomology theories:



(Scholze)

Habiro cohomology (Scholze's expectations)**.** Should have coefficients not in $\mathbb{Z}[\![q-1]\!]$, but in the *Habiro ring*

Habiro cohomology (Scholze's expectations)**.** Should have coefficients not in $\mathbb{Z}[\![q-1]\!]$, but in the *Habiro ring*

$$\mathcal{H} := \qquad \mathbb{Z}[q]$$

MAX PLANCK INSTITUTE FOR MATHEMATICS

Habiro cohomology (Scholze's expectations). Should have coefficients not in $\mathbb{Z}[\![q-1]\!]$, but in the *Habiro ring*

$$\mathcal{H} := \qquad \mathbb{Z}[q]^{\wedge}_{(q^m-1)}$$

Habiro cohomology (Scholze's expectations)**.** Should have coefficients not in $\mathbb{Z}[\![q-1]\!]$, but in the *Habiro ring*

$$\mathcal{H} := \lim_{m\in\mathbb{N}} \mathbb{Z}[q]^{\wedge}_{(q^m-1)}\,.$$

Habiro cohomology (Scholze's expectations). Should have coefficients not in $\mathbb{Z}[\![q-1]\!]$, but in the *Habiro ring*

$$\mathcal{H} := \lim_{m\in\mathbb{N}} \mathbb{Z}[q]^\wedge_{(q^m-1)}\,.$$

Moreover:

1. At $q = 1$, should get Hodge cohomology (*not* de Rham).

Habiro cohomology (Scholze's expectations)**.** Should have coefficients not in $\mathbb{Z}[\![q-1]\!]$, but in the *Habiro ring*

$$\mathcal{H} := \lim_{m\in\mathbb{N}} \mathbb{Z}[q]^{\wedge}_{(q^m-1)}\,.$$

Moreover:

1. At $q=1$, should get Hodge cohomology (*not* de Rham).
2. Should recover prismatic cohomology for all primes p.

Habiro cohomology (Scholze's expectations). Should have coefficients not in $\mathbb{Z}[\![q-1]\!]$, but in the *Habiro ring*

$$\mathcal{H} := \lim_{m\in\mathbb{N}} \mathbb{Z}[q]^\wedge_{(q^m-1)}\,.$$

Moreover:

1. At $q=1$, should get Hodge cohomology (*not* de Rham).
2. Should recover prismatic cohomology for all primes p.
3. Relation to quantum invariants of 3-manifolds?

Habiro cohomology (Scholze's expectations). Should have coefficients not in $\mathbb{Z}[\![q-1]\!]$, but in the *Habiro ring*

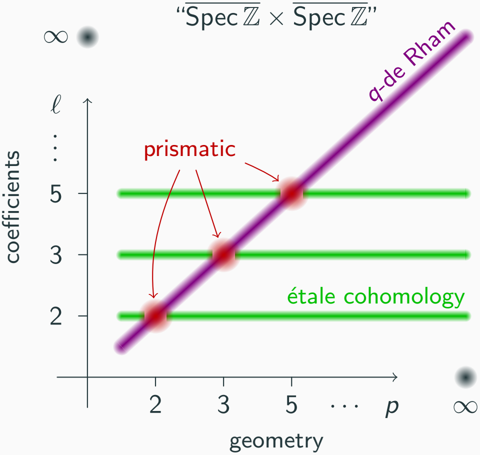
$$\mathcal{H} := \lim_{m\in\mathbb{N}} \mathbb{Z}[q]^{\wedge}_{(q^m-1)}\,.$$

Moreover:

1. At $q=1$, should get Hodge cohomology (*not* de Rham).
2. Should recover prismatic cohomology for all primes p.
3. Relation to quantum invariants of 3-manifolds? E.g. *Habiro ring of a number field* (Garoufalidis–Scholze–Wheeler–Zagier).

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, ...) by analysing the cohomology of the associated geometric objects.

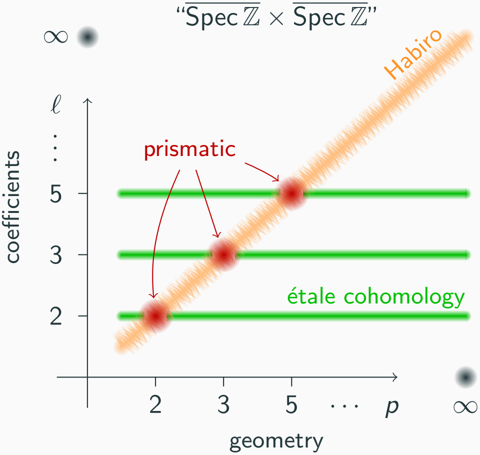
Many cohomology theories:



(Scholze)

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, ...) by analysing the cohomology of the associated geometric objects.

Many cohomology theories:

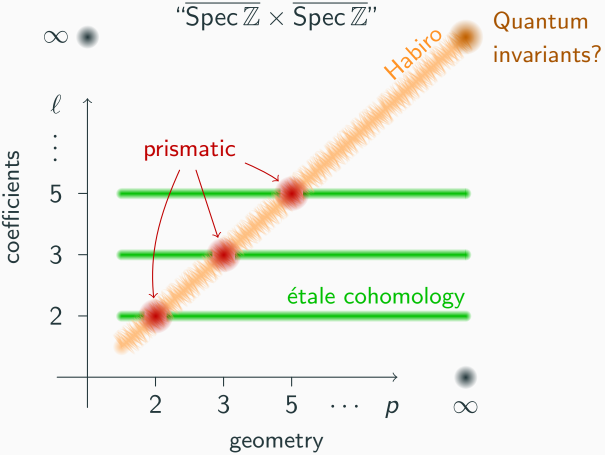


(Scholze)

MAX PLANCK INSTITUTE FOR MATHEMATICS

Study equations over $\mathbb{Z}$ or $\mathbb{Q}$ (or $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, ...) by analysing the cohomology of the associated geometric objects.

Many cohomology theories:



(Scholze)

Habiro cohomology (Scholze's expectations). Should have coefficients not in $\mathbb{Z}[\![q-1]\!]$, but in the *Habiro ring*

$$\mathcal{H} \coloneqq \lim_{m\in\mathbb{N}} \mathbb{Z}[q]^{\wedge}_{(q^m-1)}\,.$$

Moreover:

1. At $q=1$, should get Hodge cohomology (*not* de Rham).
2. Should recover prismatic cohomology for all primes p.
3. Relation to quantum invariants of 3-manifolds? E.g. *Habiro ring of a number field* (Garoufalidis–Scholze–Wheeler–Zagier).

Habiro cohomology (Scholze's expectations)**.** Should have coefficients not in $\mathbb{Z}[\![q-1]\!]$, but in the *Habiro ring*

$$\mathcal{H} \coloneqq \lim_{m \in \mathbb{N}} \mathbb{Z}[q]^{\wedge}_{(q^m-1)}\,.$$

Moreover:

1. At $q = 1$, should get Hodge cohomology (*not* de Rham).
2. Should recover prismatic cohomology for all primes p.
3. Relation to quantum invariants of 3-manifolds? E.g. *Habiro ring of a number field* (Garoufalidis–Scholze–Wheeler–Zagier).

$\Rightarrow$ **We should look for q-deformations of the Hodge filtration!**

2. q-Hodge filtrations & Habiro cohomology

q-Hodge filtrations. Let $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ be the category of pairs

$$\Big(\quad R \quad , \quad \mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R \quad \Big)$$

q-Hodge filtrations. Let $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ be the category of pairs

$$\Big(\underbrace{R}_{\text{smooth over } \mathbb{Z}} \; , \quad \mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R \quad \Big)$$

q-Hodge filtrations. Let $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ be the category of pairs

$$\Big(\underbrace{R}_{\text{smooth over } \mathbb{Z}}, \underbrace{\mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R}_{\text{filtered } (q-1)^{\star}\mathbb{Z}[\![q-1]\!]\text{-module}}\Big)$$

q-Hodge filtrations. Let $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ be the category of pairs

$$\Big(\underbrace{R}_{\text{smooth over }\mathbb{Z}},\ \underbrace{\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R}_{\text{filtered }(q-1)^\star\mathbb{Z}[\![q-1]\!]\text{-module}}\Big)$$

such that:

1. $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R/(q-1) \simeq \mathrm{fil}^\star_{\mathrm{Hdg}}\, \Omega^*_R$.

q-Hodge filtrations. Let $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ be the category of pairs

$$\Big(\underset{\text{smooth over }\mathbb{Z}}{R}\ ,\ \underset{\text{filtered }(q-1)^\star\mathbb{Z}[\![q-1]\!]\text{-module}}{\operatorname{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R}\Big)$$

such that:

1. $\operatorname{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R/(q-1) \simeq \operatorname{fil}^\star_{\mathrm{Hdg}}\,\Omega^*_R$.
2. $(\operatorname{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})^\wedge_{(q-1)} \simeq \operatorname{fil}^\star_{(\mathrm{Hdg},q-1)}(\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$

q-Hodge filtrations. Let $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ be the category of pairs

$$\Big(\underbrace{R}_{\text{smooth over }\mathbb{Z}},\ \underbrace{\mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R}_{\text{filtered }(q-1)^{\star}\mathbb{Z}[\![q-1]\!]\text{-module}}\Big)$$

such that:

1. $\mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R/(q-1) \simeq \mathrm{fil}^{\star}_{\mathrm{Hdg}}\, \Omega^*_R$.
2. $(\mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{(q-1)} \simeq \mathrm{fil}^{\star}_{(\mathrm{Hdg},q-1)}(\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$

$$\Big(\quad \text{Recall: } (q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{(q-1)} \simeq (\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]. \quad\Big)$$

q-Hodge filtrations. Let $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ be the category of pairs

$$\Big(\underset{\text{smooth over }\mathbb{Z}}{R}\ ,\ \underset{\text{filtered }(q-1)^\star\mathbb{Z}[\![q-1]\!]\text{-module}}{\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R}\Big)$$

such that:

1. $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R/(q-1) \simeq \mathrm{fil}^\star_{\mathrm{Hdg}}\, \Omega^*_R$.
2. $(\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})^\wedge_{(q-1)} \simeq \mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}(\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$

$$\Big(\qquad \text{Recall: } (q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})^\wedge_{(q-1)} \simeq (\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]. \qquad\Big)$$

q-Hodge filtrations. Let $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ be the category of pairs

$$\Big(\underbrace{R}_{\text{smooth over }\mathbb{Z}},\ \underbrace{\mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R}_{\text{filtered }(q-1)^{\star}\mathbb{Z}[\![q-1]\!]\text{-module}}\Big)$$

such that:

1. $\mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R/(q-1) \simeq \mathrm{fil}^{\star}_{\mathrm{Hdg}}\,\Omega^*_R$.
2. $(\mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{(q-1)} \simeq \mathrm{fil}^{\star}_{(\mathrm{Hdg},q-1)}(\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$

$$\Big(\quad \text{Recall: } (q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{(q-1)} \simeq (\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]. \quad\Big)$$

q-Hodge filtrations. Let $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ be the category of pairs

$$\Big(\underbrace{R}_{\text{smooth over }\mathbb{Z}}, \underbrace{\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R}_{\text{filtered }(q-1)^\star\mathbb{Z}[\![q-1]\!]\text{-module}}\Big)$$

such that:

1. $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R/(q-1) \simeq \mathrm{fil}^\star_{\mathrm{Hdg}}\, \Omega^*_R$.
2. $(\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})^\wedge_{(q-1)} \simeq \mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}(\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]\,,$
   $\mathrm{fil}^\star_{q\text{-Hdg}}(q\text{-}\Omega_R)^\wedge_p\left[\tfrac{1}{p}\right]^\wedge_{(q-1)} \simeq \mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}(\Omega^*_R)^\wedge_p\left[\tfrac{1}{p}\right][\![q-1]\!]$ for all $p\,.$

q-Hodge filtrations. Let $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ be the category of pairs

$$\Big(\underset{\text{smooth over } \mathbb{Z}}{R} \;,\; \underset{\text{filtered } (q-1)^\star\mathbb{Z}[\![q-1]\!]\text{-module}}{\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R} \Big)$$

such that:

1. $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R/(q-1) \simeq \mathrm{fil}^\star_{\mathrm{Hdg}}\, \Omega^*_R$.
2. $(\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})^\wedge_{(q-1)} \simeq \mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}(\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$,

   $\mathrm{fil}^\star_{q\text{-Hdg}}(q\text{-}\Omega_R)^\wedge_p\left[\tfrac{1}{p}\right]^\wedge_{(q-1)} \simeq \mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}(\Omega^*_R)^\wedge_p\left[\tfrac{1}{p}\right][\![q-1]\!]$ for all p.

Theorem (W. 2025).

The forgetful functor $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}} \to \mathrm{Sm}_{\mathbb{Z}}$ has *no* section

q-Hodge filtrations. Let $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ be the category of pairs

$$\Bigl(\underset{\text{smooth over }\mathbb{Z}}{R}\ ,\ \underset{\text{filtered }(q-1)^{\star}\mathbb{Z}[\![q-1]\!]\text{-module}}{\mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R}\Bigr)$$

such that:

1. $\mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R/(q-1) \simeq \mathrm{fil}^{\star}_{\mathrm{Hdg}}\,\Omega^*_R$.
2. $(\mathrm{fil}^{\star}_{q\text{-Hdg}}\, q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{(q-1)} \simeq \mathrm{fil}^{\star}_{(\mathrm{Hdg},q-1)}(\Omega^*_R \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]\,,$
   $\mathrm{fil}^{\star}_{q\text{-Hdg}}(q\text{-}\Omega_R)^{\wedge}_p\bigl[\tfrac{1}{p}\bigr]^{\wedge}_{(q-1)} \simeq \mathrm{fil}^{\star}_{(\mathrm{Hdg},q-1)}(\Omega^*_R)^{\wedge}_p\bigl[\tfrac{1}{p}\bigr][\![q-1]\!]$ for all p.

Theorem (W. 2025).

The forgetful functor $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}} \to \mathrm{Sm}_{\mathbb{Z}}$ has *no* section, but over the full subcategory

$$\bigl\{R \bigm| p \in R^{\times} \text{ for all primes } p \leqslant \dim(R/\mathbb{Z})\bigr\} \subseteq \mathrm{Sm}_{\mathbb{Z}}$$

there exists a section.

The q-Hodge complex. Given $(R, \mathrm{fil}^{\star}_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_R)$, define the *$q$-Hodge complex*

$$q\text{-}\mathrm{Hdg}_R :=$$

The q-Hodge complex. Given $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R)$, define the *$q$-Hodge complex*

$$q\text{-Hdg}_R \coloneqq q\text{-}\Omega_R$$

The q-Hodge complex. Given $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R)$, define the *$q$-Hodge complex*

$$q\text{-Hdg}_R \coloneqq q\text{-}\Omega_R\left[\frac{\mathrm{fil}^n_{q\text{-Hdg}}}{(q-1)^n}\;\middle|\; n \geqslant 1\right]^\wedge_{(q-1)}.$$

The q-Hodge complex. Given $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R)$, define the *$q$-Hodge complex*

$$q\text{-}\mathrm{Hdg}_R \coloneqq q\text{-}\Omega_R \left[\frac{\mathrm{fil}^n_{q\text{-Hdg}}}{(q-1)^n} \;\middle|\; n \geqslant 1 \right]^\wedge_{(q-1)} .$$

The q-Hodge complex. Given $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R)$, define the *$q$-Hodge complex*

$$q\text{-Hdg}_R \coloneqq q\text{-}\Omega_R\left[\frac{\mathrm{fil}^n_{q\text{-Hdg}}}{(q-1)^n}\;\middle|\; n \geqslant 1\right]^\wedge_{(q-1)} .$$

Theorem (W. 2025).

1 There exists a non-trivial factorisation

$$\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}} \xrightarrow[q\text{-Hdg}_{(-)}]{} \mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big)$$

The q-Hodge complex. Given $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R)$, define the *$q$-Hodge complex*

$$q\text{-Hdg}_R \coloneqq q\text{-}\Omega_R\left[\frac{\mathrm{fil}^n_{q\text{-Hdg}}}{(q-1)^n}\;\middle|\; n \geqslant 1\right]^\wedge_{(q-1)}.$$

Theorem (W. 2025).

1 There exists a non-trivial factorisation

$$\begin{tikzcd} & \mathcal{D}(\mathcal{H}) \arrow[d, "(-)^\wedge_{(q-1)}"] \\ \mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}} \arrow[r, "q\text{-Hdg}_{(-)}"'] & \mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big) \end{tikzcd}$$

The q-Hodge complex. Given $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R)$, define the *$q$-Hodge complex*

$$q\text{-Hdg}_R \coloneqq q\text{-}\Omega_R\left[\frac{\mathrm{fil}^n_{q\text{-Hdg}}}{(q-1)^n}\ \middle|\ n \geqslant 1\right]^\wedge_{(q-1)}.$$

Theorem (W. 2025).

1. There exists a non-trivial factorisation

$$\begin{tikzcd} & \mathcal{D}(\mathcal{H}) \arrow[d, "(-)^\wedge_{(q-1)}"] \\ \mathrm{Sm}^{q\text{-Hdg}}_{\mathbb{Z}} \arrow[ur, dashed, "\text{Habiro–Hodge complex } q\text{-}\mathcal{H}\mathrm{dg}_{(-)}"] \arrow[r, "q\text{-Hdg}_{(-)}"'] & \mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big) \end{tikzcd}$$

The q-Hodge complex. Given $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R)$, define the *$q$-Hodge complex*

$$q\text{-Hdg}_R \coloneqq q\text{-}\Omega_R\left[\frac{\mathrm{fil}^n_{q\text{-Hdg}}}{(q-1)^n}\;\middle|\; n \geqslant 1\right]^\wedge_{(q-1)}.$$

Theorem (W. 2025).

1. There exists a non-trivial factorisation

$$\begin{tikzcd} & \mathcal{D}(\mathcal{H}) \arrow[d, "(-)^\wedge_{(q-1)}"] \\ \mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}} \arrow[ur, dashed, "\textit{Habiro–Hodge complex}\ q\text{-}\mathcal{H}\mathrm{dg}_{(-)}"] \arrow[r, "q\text{-Hdg}_{(-)}"'] & \mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big) \end{tikzcd}$$

2. $\mathrm{H}^*\big(q\text{-}\mathcal{H}\mathrm{dg}_R/(q^m-1)\big) \cong q\text{-}\mathbb{W}_m\Omega^*_R$.

The q-Hodge complex. Given $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R)$, define the *$q$-Hodge complex*

$$q\text{-}\mathrm{Hdg}_R \coloneqq q\text{-}\Omega_R\left[\frac{\mathrm{fil}^n_{q\text{-Hdg}}}{(q-1)^n}\;\middle|\; n \geqslant 1\right]^\wedge_{(q-1)}.$$

Theorem (W. 2025).

1. There exists a non-trivial factorisation

$$\begin{tikzcd} & \mathcal{D}(\mathcal{H}) \arrow[d, "(-)^\wedge_{(q-1)}"] \\ \mathrm{Sm}^{q\text{-}\mathrm{Hdg}}_{\mathbb{Z}} \arrow[ur, dashed, "\textit{Habiro–Hodge complex}\; q\text{-}\mathcal{H}\mathrm{dg}_{(-)}"] \arrow[r, "q\text{-}\mathrm{Hdg}_{(-)}"'] & \mathcal{D}\big(\mathbb{Z}[\![q-1]\!]\big) \end{tikzcd}$$

2. $\mathrm{H}^*\big(q\text{-}\mathcal{H}\mathrm{dg}_R/(q^m-1)\big) \cong q\text{-}\mathbb{W}_m\Omega^*_R$. In particular,

$$\mathrm{H}^*\big(q\text{-}\mathcal{H}\mathrm{dg}_R/(q-1)\big) \cong \Omega^*_R\,.$$

Theorem (continued)

3. $q\text{-}\Omega_R \simeq \mathrm{L}\eta_{(q-1)}(q\text{-}\mathrm{Hdg}_R)$.

Theorem (continued)

3. $q\text{-}\Omega_R \simeq \mathrm{L}\eta_{(q-1)}(q\text{-}\mathrm{Hdg}_R)$.
4. The *Habiro ring of a number field F* (Garoufalidis–Scholze–Wheeler–Zagier) can be recovered as

$$q\text{-}\mathcal{H}\mathrm{dg}_{\mathcal{O}_F[1/\Delta]} \simeq \mathcal{H}_{\mathcal{O}_F[1/\Delta]}$$

($\Delta =$ discriminant of F).

Theorem (continued)

3. $q\text{-}\Omega_R \simeq \mathrm{L}\eta_{(q-1)}(q\text{-}\mathrm{Hdg}_R)$.
4. The *Habiro ring of a number field F* (Garoufalidis–Scholze–Wheeler–Zagier) can be recovered as

$$q\text{-}\mathcal{H}\mathrm{dg}_{\mathcal{O}_F[1/\Delta]} \simeq \mathcal{H}_{\mathcal{O}_F[1/\Delta]}$$

($\Delta =$ discriminant of F).

If X is a smooth scheme over $\mathbb{Z}$, up to inverting all primes $p \leqslant \dim(X/\mathbb{Z})$, one can define *algebraic Habiro cohomology of X* as

$$\mathrm{R}\Gamma_{\mathcal{H}}(X) \coloneqq \mathrm{R}\Gamma(X, q\text{-}\mathcal{H}\mathrm{dg}_X)\,.$$

Theorem (continued)

3. $q\text{-}\Omega_R \simeq \mathrm{L}\eta_{(q-1)}(q\text{-}\mathrm{Hdg}_R)$.
4. The *Habiro ring of a number field F* (Garoufalidis–Scholze–Wheeler–Zagier) can be recovered as

$$q\text{-}\mathcal{H}\mathrm{dg}_{\mathcal{O}_F[1/\Delta]} \simeq \mathcal{H}_{\mathcal{O}_F[1/\Delta]}$$

($\Delta =$ discriminant of F).

If X is a smooth scheme over $\mathbb{Z}$, up to inverting all primes $p \leqslant \dim(X/\mathbb{Z})$, one can define *algebraic Habiro cohomology of X* as

$$\mathrm{R}\Gamma_{\mathcal{H}}(X) \coloneqq \mathrm{R}\Gamma(X, q\text{-}\mathcal{H}\mathrm{dg}_X)\,.$$

⚠ Probably no six functor formalism/stacky approach.

Theorem (continued)

3. $q\text{-}\Omega_R \simeq \mathrm{L}\eta_{(q-1)}(q\text{-}\mathrm{Hdg}_R)$.
4. The *Habiro ring of a number field F* (Garoufalidis–Scholze–Wheeler–Zagier) can be recovered as

$$q\text{-}\mathcal{H}\mathrm{dg}_{\mathcal{O}_F[1/\Delta]} \simeq \mathcal{H}_{\mathcal{O}_F[1/\Delta]}$$

($\Delta =$ discriminant of F).

If X is a smooth scheme over $\mathbb{Z}$, up to inverting all primes $p \leqslant \dim(X/\mathbb{Z})$, one can define *algebraic Habiro cohomology of X* as

$$\mathrm{R}\Gamma_{\mathcal{H}}(X) \coloneqq \mathrm{R}\Gamma(X, q\text{-}\mathcal{H}\mathrm{dg}_X)\,.$$

⚠ Probably no six functor formalism/stacky approach. But there should be cohomology classes from quantum invariants (Garoufalidis–Wheeler).

3. q-Hodge filtrations from homotopy theory

Recall: For an associative ring R, *Hochschild homology of R* is

$$\mathrm{HH}(R) := \left| \qquad \text{cyclic bar construction} \qquad \right| .$$

Recall: For an associative ring R, *Hochschild homology of R* is

$$\mathrm{HH}(R) \coloneqq \left| \, R \leftleftarrows R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \; \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} \; R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \; \overset{\longleftarrow}{\underset{\longleftarrow}{\overset{\longleftarrow}{\underset{\longleftarrow}{}}}} \cdots \, \right| .$$

Recall: For an associative ring R, *Hochschild homology of R* is

$$\mathrm{HH}(R) \coloneqq \left|\, R \leftleftarrows R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \; \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} \; R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \; \overset{\longleftarrow}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longleftarrow}}} \; \cdots \,\right|.$$

There's an action $S^1 \curvearrowright \mathrm{HH}(R)$, put $\mathrm{HC}^-(R) \coloneqq \mathrm{HH}(R)^{hS^1}$ (Connes).

Recall: For an associative ring R, *Hochschild homology of R* is

$$\mathrm{HH}(R) := \left| \; R \leftleftarrows R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \;\overset{\longleftarrow}{\leftleftarrows}\; R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \;\overset{\leftleftarrows}{\leftleftarrows}\; \cdots \; \right| .$$

There's an action $S^1 \curvearrowright \mathrm{HH}(R)$, put $\mathrm{HC}^-(R) := \mathrm{HH}(R)^{hS^1}$ (Connes).

Theorem (…, Antieau)**.**

Recall: For an associative ring R, *Hochschild homology of R* is

$$\mathrm{HH}(R) \coloneqq \left| R \leftleftarrows R \otimes^{\mathbb{L}}_{\mathbb{Z}} R \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} R \otimes^{\mathbb{L}}_{\mathbb{Z}} R \otimes^{\mathbb{L}}_{\mathbb{Z}} R \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}} \cdots \right| .$$

There's an action $S^1 \curvearrowright \mathrm{HH}(R)$, put $\mathrm{HC}^-(R) \coloneqq \mathrm{HH}(R)^{hS^1}$ (Connes).

Theorem (…, Antieau)**.**

If R is quasisyntomic, there is a complete exhaustive "motivic" filtration $\mathrm{fil}^{\star}_{\mathrm{mot}}\,\mathrm{HC}^-(R)$ with associated graded

$$\mathrm{gr}^i_{\mathrm{mot}}\,\mathrm{HC}^-(R) \simeq \mathrm{fil}^i_{\mathrm{Hdg}}\,\mathrm{dR}_R$$

Recall: For an associative ring R, *Hochschild homology of R* is

$$\mathrm{HH}(R) := \left| R \leftleftarrows R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \;\underleftarrow{\leftleftarrows}\; R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \;\leftleftarrows\!\!\leftleftarrows \cdots \right| .$$

There's an action $S^1 \curvearrowright \mathrm{HH}(R)$, put $\mathrm{HC}^-(R) := \mathrm{HH}(R)^{hS^1}$ (Connes).

Theorem (..., Antieau).

If R is quasisyntomic, there is a complete exhaustive "motivic" filtration $\mathrm{fil}^\star_{\mathrm{mot}}\, \mathrm{HC}^-(R)$ with associated graded

$$\mathrm{gr}^i_{\mathrm{mot}}\, \mathrm{HC}^-(R) \simeq \mathrm{fil}^i_{\mathrm{Hdg}}\, \mathrm{dR}_R$$

(where $\mathrm{dR}_{(-)}$ denotes the non-abelian derived functor of $\Omega^*_{(-)}$).

Recall: For an associative ring R, *Hochschild homology of R* is

$$\mathrm{HH}(R) \coloneqq \left| \, R \leftleftarrows R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \,\overset{\leftarrow}{\leftleftarrows}\, R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \otimes^{\mathsf{L}}_{\mathbb{Z}} R \,\overset{\leftleftarrows}{\leftleftarrows}\, \cdots \, \right| .$$

There's an action $S^1 \curvearrowright \mathrm{HH}(R)$, put $\mathrm{HC}^-(R) \coloneqq \mathrm{HH}(R)^{hS^1}$ (Connes).

Theorem (. . . , Antieau)**.**

If R is quasisyntomic, there is a complete exhaustive "motivic" filtration $\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{HC}^-(R)$ with associated graded

$$\mathrm{gr}^i_{\mathrm{mot}} \mathrm{HC}^-(R) \simeq \mathrm{fil}^i_{\mathrm{Hdg}} \widehat{\mathrm{dR}}_R[2i]$$

(where $\mathrm{dR}_{(-)}$ denotes the non-abelian derived functor of $\Omega^*_{(-)}$).

Even filtration (Hahn–Raksit–Wilson)**.** The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

Even filtration (Hahn–Raksit–Wilson)**.** The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^{-}(R)$:

$$\mathrm{fil}^{\star}_{\mathrm{ev}}\ \mathrm{HH}(R) \quad := \lim_{\substack{\mathrm{HH}(R)\to E \text{ even} \\ \mathbb{E}_\infty}} \tau_{\geqslant 2\star}(\ E\ )\,.$$

Even filtration (Hahn–Raksit–Wilson)**.** The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

$$\mathrm{fil}^\star_{\mathrm{ev}}\ \mathrm{HH}(R) \quad := \lim_{\substack{\mathrm{HH}(R)\to E\ \text{even} \\ \mathbb{E}_\infty}} \tau_{\geqslant 2\star}(\ E\ )\,.$$

Even filtration (Hahn–Raksit–Wilson)**.** The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

$$\mathrm{fil}^{\star}_{\mathrm{ev},hS^1}\,\mathrm{HC}^-(R) \coloneqq \lim_{\substack{\mathrm{HH}(R)\to E \text{ even} \\ S^1 \text{ equivariant } \mathbb{E}_\infty}} \tau_{\geqslant 2\star}\left(E^{hS^1}\right).$$

Even filtration (Hahn–Raksit–Wilson)**.** The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

$$\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{HC}^-(R) \coloneqq \lim_{\substack{\mathrm{HH}(R)\to E \text{ even} \\ S^1 \text{ equivariant } \mathbb{E}_\infty}} \tau_{\geqslant 2\star}\left(E^{hS^1}\right).$$

The even filtration works equally well for $\mathrm{HC}^-(-/\mathrm{ku})$.

Even filtration (Hahn–Raksit–Wilson)**.** The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

$$\mathrm{fil}^\star_{\mathrm{ev},hS^1}\mathrm{HC}^-(R) \coloneqq \lim_{\substack{\mathrm{HH}(R)\to E \text{ even} \\ S^1 \text{ equivariant } \mathbb{E}_\infty}} \tau_{\geqslant 2\star}\left(E^{hS^1}\right).$$

The even filtration works equally well for $\mathrm{HC}^-(-/\mathrm{ku})$. connective complex K-theory

Even filtration (Hahn–Raksit–Wilson)**.** The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

$$\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{HC}^-(R) \coloneqq \lim_{\substack{\mathrm{HH}(R)\to E\text{ even}\\ S^1\text{ equivariant }\mathbb{E}_\infty}} \tau_{\geqslant 2\star}\big(E^{hS^1}\big)\,.$$

The even filtration works equally well for $\mathrm{TC}^-(-/\underbrace{\mathrm{ku}}^{\text{connective complex }K\text{-theory}})$.

Even filtration (Hahn–Raksit–Wilson)**.** The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

$$\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{HC}^-(R) \coloneqq \lim_{\substack{\mathrm{HH}(R)\to E \text{ even} \\ S^1 \text{ equivariant } \mathbb{E}_\infty}} \tau_{\geqslant 2\star}\left(E^{hS^1}\right).$$

connective complex K-theory

The even filtration works equally well for $\mathrm{TC}^-(-/\mathrm{ku})$.
There's a version for $\mathbb{E}_1$-ring spectra too (Pstrągowski).

$\mathrm{TC}^-(-/\mathrm{ku})$ & q-Hodge filtrations

Even filtration (Hahn–Raksit–Wilson)**.** The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

$$\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{HC}^-(R) \coloneqq \lim_{\substack{\mathrm{HH}(R)\to E\text{ even}\\ S^1\text{ equivariant }\mathbb{E}_\infty}} \tau_{\geqslant 2\star}\big(E^{hS^1}\big)\,.$$

The even filtration works equally well for $\mathrm{TC}^-(-/\overset{\text{connective complex }K\text{-theory}}{\mathrm{ku}})$.
There's a version for $\mathbb{E}_1$-ring spectra too (Pstrągowski).

Theorem (W. 2025)**.**

Let R be quasisyntomic,

(Raksit proved this first for $\mathbb{S}[x]$; closely related work of Devalapurkar)

Even filtration (Hahn–Raksit–Wilson). The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

$$\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{HC}^-(R) \coloneqq \lim_{\substack{\mathrm{HH}(R)\to E \text{ even} \\ S^1 \text{ equivariant } \mathbb{E}_\infty}} \tau_{\geqslant 2\star}\big(E^{hS^1}\big)\,.$$

The even filtration works equally well for $\mathrm{TC}^-(-/\underbrace{\mathrm{ku}}^{\text{connective complex } K\text{-theory}})$.

There's a version for $\mathbb{E}_1$-ring spectra too (Pstrągowski).

Theorem (W. 2025).

Let R be quasisyntomic, $2 \in R^\times$.

(Raksit proved this first for $\mathbb{S}[x]$; closely related work of Devalapurkar)

Even filtration (Hahn–Raksit–Wilson). The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

$$\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{HC}^-(R) \coloneqq \lim_{\substack{\mathrm{HH}(R)\to E \text{ even} \\ S^1 \text{ equivariant } \mathbb{E}_\infty}} \tau_{\geqslant 2\star}\big(E^{hS^1}\big)\,.$$

The even filtration works equally well for $\mathrm{TC}^-(-/\underbrace{\mathrm{ku}}^{\text{connective complex } K\text{-theory}})$.
There's a version for $\mathbb{E}_1$-ring spectra too (Pstrągowski).

Theorem (W. 2025).

Let R be quasisyntomic, $2 \in R^\times$. Suppose R lifts to an $\mathbb{E}_2$-ring spectrum $\mathbb{S}_R$ such that $\mathbb{S}_R \otimes \mathbb{Z} \simeq R$.

(Raksit proved this first for $\mathbb{S}[x]$; closely related work of Devalapurkar)

Even filtration (Hahn–Raksit–Wilson). The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

$$\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{HC}^-(R) \coloneqq \lim_{\substack{\mathrm{HH}(R)\to E \text{ even} \\ S^1 \text{ equivariant } \mathbb{E}_\infty}} \tau_{\geqslant 2\star}\big(E^{hS^1}\big)\,.$$

The even filtration works equally well for $\mathrm{TC}^-(-/\overset{\text{connective complex } K\text{-theory}}{\mathrm{ku}})$.
There's a version for $\mathbb{E}_1$-ring spectra too (Pstrągowski).

Theorem (W. 2025).

Let R be quasisyntomic, $2 \in R^\times$. Suppose R lifts to an $\mathbb{E}_2$-ring spectrum $\mathbb{S}_R$ such that $\mathbb{S}_R \otimes \mathbb{Z} \simeq R$. Then there exists a q-Hodge filtration $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_R$ such that

$$\mathrm{gr}^i_{\mathrm{ev},hS^1}\,\mathrm{TC}^-(\mathrm{ku}\otimes\mathbb{S}_R/\mathrm{ku}) \simeq \mathrm{fil}^i_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_R \quad .$$

(Raksit proved this first for $\mathbb{S}[x]$; closely related work of Devalapurkar)

Even filtration (Hahn–Raksit–Wilson)**.** The motivic filtration agrees with the *even filtration* on $\mathrm{HC}^-(R)$:

$$\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{HC}^-(R) \coloneqq \lim_{\substack{\mathrm{HH}(R)\to E \text{ even}\\ S^1 \text{ equivariant } \mathbb{E}_\infty}} \tau_{\geqslant 2\star}(E^{hS^1})\,.$$

The even filtration works equally well for $\mathrm{TC}^-(-/\overset{\text{connective complex } K\text{-theory}}{\mathrm{ku}})$.
There's a version for $\mathbb{E}_1$-ring spectra too (Pstrągowski).

Theorem (W. 2025)**.**

Let R be quasisyntomic, $2 \in R^\times$. Suppose R lifts to an $\mathbb{E}_2$-ring spectrum $\mathbb{S}_R$ such that $\mathbb{S}_R \otimes \mathbb{Z} \simeq R$. Then there exists a q-Hodge filtration $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_R$ such that

$$\mathrm{gr}^i_{\mathrm{ev},hS^1}\,\mathrm{TC}^-(\mathrm{ku}\otimes\mathbb{S}_R/\mathrm{ku}) \simeq \mathrm{fil}^i_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_R[2i]\,.$$

(Raksit proved this first for $\mathbb{S}[x]$; closely related work of Devalapurkar)

Theorem (Antieau, Hahn–Raksit–Wilson).

Let R be quasisyntomic. Then

$$\mathrm{gr}^i_{\mathrm{ev},hS^1}\,\mathrm{HC}^-(R) \simeq \mathrm{fil}^i_{\mathrm{Hdg}}\,\widehat{\mathrm{dR}}_R[2i]\,.$$

$$\uparrow$$

deformation $\mathrm{ku} \to \mathbb{Z}$ | deformation $q\text{-}\mathrm{dR}_R \to \mathrm{dR}_R$

Theorem (W. 2025).

Let R be quasisyntomic, $2 \in R^\times$. Suppose R lifts to an $\mathbb{E}_2$-ring spectrum $\mathbb{S}_R$ such that $\mathbb{S}_R \otimes \mathbb{Z} \simeq R$. Then there exists a q-Hodge filtration $\mathrm{fil}^\star_{q\text{-Hdg}}\,q\text{-}\mathrm{dR}_R$ such that

$$\mathrm{gr}^i_{\mathrm{ev},hS^1}\,\mathrm{TC}^-(\mathrm{ku}\otimes\mathbb{S}_R/\mathrm{ku}) \simeq \mathrm{fil}^i_{q\text{-Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_R[2i]\,.$$

(Raksit proved this first for $\mathbb{S}[x]$; closely related work of Devalapurkar)

Theorem (W. 2025)**.**

Let $\mathrm{KU} := \mathrm{ku}[\beta^{-1}]$ be the periodic complex K-theory spectrum.

Theorem (W. 2025)**.**

Let $\mathrm{KU} \coloneqq \mathrm{ku}[\beta^{-1}]$ be the periodic complex K-theory spectrum. Then

$$\mathrm{gr}^0_{\mathrm{ev},hS^1}\,\mathrm{TC}^-(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU}) \simeq q\text{-}\mathrm{Hdg}_R\ .$$

Theorem (W. 2025).

Let $\mathrm{KU} \coloneqq \mathrm{ku}[\beta^{-1}]$ be the periodic complex K-theory spectrum. Then

$$\mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU}) \simeq q\text{-}\mathrm{Hdg}_R \ .$$

The Habiro–Hodge complex is given by

$$\mathrm{THH}(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU})$$

Theorem (W. 2025).

Let $\mathrm{KU} \coloneqq \mathrm{ku}[\beta^{-1}]$ be the periodic complex K-theory spectrum. Then

$$\mathrm{gr}^0_{\mathrm{ev},hS^1}\,\mathrm{TC}^-(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU}) \simeq q\text{-}\mathrm{Hdg}_R\ .$$

The Habiro–Hodge complex is given by

$$\mathrm{THH}(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU})^{C_m} \qquad ,$$

where $(-)^{C_m}$ denotes genuine fixed points for $C_m \subseteq S^1$.

Theorem (W. 2025)**.**

Let $\mathrm{KU} \coloneqq \mathrm{ku}[\beta^{-1}]$ be the periodic complex K-theory spectrum. Then

$$\mathrm{gr}^0_{\mathrm{ev},hS^1}\,\mathrm{TC}^-(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU}) \simeq q\text{-}\mathrm{Hdg}_R\ .$$

The Habiro–Hodge complex is given by

$$\left(\mathrm{THH}(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU})^{C_m}\right)^{h(S^1/C_m)} \qquad ,$$

where $(-)^{C_m}$ denotes genuine fixed points for $C_m \subseteq S^1$.

MAX PLANCK INSTITUTE FOR MATHEMATICS

Theorem (W. 2025).

Let $\mathrm{KU} := \mathrm{ku}[\beta^{-1}]$ be the periodic complex K-theory spectrum. Then

$$\mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU}) \simeq q\text{-}\mathrm{Hdg}_R \,.$$

The Habiro–Hodge complex is given by

$$\lim_{m \in \mathbb{N}} \left(\mathrm{THH}(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU})^{C_m}\right)^{h(S^1/C_m)} \qquad ,$$

where $(-)^{C_m}$ denotes genuine fixed points for $C_m \subseteq S^1$.

MAX PLANCK INSTITUTE FOR MATHEMATICS

Theorem (W. 2025)**.**

Let $\mathrm{KU} := \mathrm{ku}[\beta^{-1}]$ be the periodic complex K-theory spectrum. Then

$$\mathrm{gr}^0_{\mathrm{ev},hS^1}\,\mathrm{TC}^-(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU})\simeq q\text{-}\mathrm{Hdg}_R\,.$$

The Habiro–Hodge complex is given by

$$\mathrm{gr}^0_{\mathrm{ev},S^1}\left(\lim_{m\in\mathbb{N}}\left(\mathrm{THH}(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU})^{C_m}\right)^{h(S^1/C_m)}\right)\simeq q\text{-}\mathcal{H}\mathrm{dg}_R,$$

where $(-)^{C_m}$ denotes genuine fixed points for $C_m\subseteq S^1$.

Theorem (W. 2025).

Let $\mathrm{KU} \coloneqq \mathrm{ku}[\beta^{-1}]$ be the periodic complex K-theory spectrum. Then

$$\mathrm{gr}^0_{\mathrm{ev},hS^1}\,\mathrm{TC}^-(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU}) \simeq q\text{-}\mathrm{Hdg}_R\ .$$

The Habiro–Hodge complex is given by

$$\mathrm{gr}^0_{\mathrm{ev},S^1}\Bigg(\underbrace{\lim_{m\in\mathbb{N}}\Big(\mathrm{THH}(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU})^{C_m}\Big)^{h(S^1/C_m)}}_{\textit{topological cyclonic homology}\ \mathrm{TCn}(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU})}\Bigg) \simeq q\text{-}\mathcal{H}\mathrm{dg}_R,$$

where $(-)^{C_m}$ denotes genuine fixed points for $C_m \subseteq S^1$.

4. Refined THH/TC$^-$ & analytic Habiro cohomology

⚠ **Problems:**

Problems:

1. To define algebraic Habiro cohomology, need to invert "small" primes or an $\mathbb{E}_2$-lift to the sphere spectrum.

Habiro cohomology over $\mathbb{Q}$?

⚠ **Problems:**

1. To define algebraic Habiro cohomology, need to invert "small" primes or an $\mathbb{E}_2$-lift to the sphere spectrum.
"Solution": Work with varieties over $\mathbb{Q}$.

Problems:

1. To define algebraic Habiro cohomology, need to invert "small" primes or an $\mathbb{E}_2$-lift to the sphere spectrum.
   "Solution": Work with varieties over $\mathbb{Q}$.
2. Recall:

$$\left(q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q}\right)_{(q-1)} \simeq \left(\Omega_R^* \otimes_{\mathbb{Z}} \mathbb{Q}\right)[\![q-1]\!] \,.$$

Problems:

1. To define algebraic Habiro cohomology, need to invert "small" primes or an $\mathbb{E}_2$-lift to the sphere spectrum.
   "Solution": Work with varieties over $\mathbb{Q}$.
2. Recall:
   $$(q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})_{(q-1)} \simeq (\Omega_R^* \otimes_{\mathbb{Z}} \mathbb{Q})\llbracket q-1 \rrbracket \,.$$
   So for varieties over $\mathbb{Q}$, we only get de Rham information, no comparison maps to étale/singular cohomology with torsion coefficients.

⚠ **Problems:**

1. To define algebraic Habiro cohomology, need to invert "small" primes or an $\mathbb{E}_2$-lift to the sphere spectrum.
   "Solution": Work with varieties over $\mathbb{Q}$.

2. Recall:
$$(q\text{-}\Omega_R \otimes_{\mathbb{Z}} \mathbb{Q})_{(q-1)} \simeq (\Omega_R^* \otimes_{\mathbb{Z}} \mathbb{Q}) \llbracket q-1 \rrbracket\,.$$
   So for varieties over $\mathbb{Q}$, we only get de Rham information, no comparison maps to étale/singular cohomology with torsion coefficients.
   Solution: Refined THH/TC$^-$.

Construction (Efimov–Scholze)**.** Let k be an $\mathbb{E}_\infty$-ring spectrum.

Construction (Efimov–Scholze)**.** Let k be an $\mathbb{E}_\infty$-ring spectrum.

$$\mathrm{Mot}_k^{\mathrm{loc}} \xrightarrow[\otimes]{\mathrm{THH}(-/k)} \mathrm{Mod}_k^{\mathrm{B}S^1}$$

Construction (Efimov–Scholze). Let k be an $\mathbb{E}_\infty$-ring spectrum.

$$\overset{\text{rigid}}{\mathrm{Mot}_k^{\mathrm{loc}}} \xrightarrow[\otimes]{\mathrm{THH}(-/k)} \mathrm{Mod}_k^{\mathrm{B}S^1}$$

Construction (Efimov–Scholze)**.** Let k be an $\mathbb{E}_\infty$-ring spectrum.

$$\overset{\text{rigid}}{\mathrm{Mot}_k^{\mathrm{loc}}} \xrightarrow[\otimes]{\mathrm{THH}(-/k)} \overset{\text{not rigid}}{\mathrm{Mod}_k^{BS^1}}$$

Construction (Efimov–Scholze)**.** Let k be an $\mathbb{E}_\infty$-ring spectrum.

$$\begin{tikzcd}
\overset{\text{rigid}}{\mathrm{Mot}_k^{\mathrm{loc}}} \arrow[r, "\mathrm{THH}(-/k)", "\otimes"'] \arrow[dr, dashed, "\mathrm{THH}^{\mathrm{ref}}(-/k)"'] & \overset{\text{not rigid}}{\mathrm{Mod}_k^{\mathrm{B}S^1}} \\
& \left(\mathrm{Mod}_k^{\mathrm{B}S^1}\right)^{\mathrm{rig}} \arrow[u]
\end{tikzcd}$$

Construction (Efimov–Scholze)**.** Let k be an $\mathbb{E}_\infty$-ring spectrum. Suppose k is complex orientable, choose complex orientation $t \in \pi_{-2}(k^{hS^1})$.

$$\begin{tikzcd}
\overset{\text{rigid}}{\mathrm{Mot}_k^{\mathrm{loc}}} \arrow[r, "\mathrm{THH}(-/k)", "\otimes"'] \arrow[dr, dashed, "\mathrm{THH}^{\mathrm{ref}}(-/k)"'] & \overset{\text{not rigid}}{\mathrm{Mod}_k^{\mathrm{B}S^1}} \\
 & \left(\mathrm{Mod}_k^{\mathrm{B}S^1}\right)^{\mathrm{rig}} \arrow[u]
\end{tikzcd}$$

Construction (Efimov–Scholze). Let k be an $\mathbb{E}_\infty$-ring spectrum. Suppose k is complex orientable, choose complex orientation $t \in \pi_{-2}(k^{hS^1})$.

$$\begin{tikzcd}
\overset{\text{rigid}}{\mathrm{Mot}_k^{\mathrm{loc}}} \arrow[r, "\mathrm{THH}(-/k)", "\otimes"'] \arrow[dr, dashed, "\mathrm{THH}^{\mathrm{ref}}(-/k)"'] & \overset{\text{not rigid}}{\mathrm{Mod}_k^{\mathrm{B}S^1}} \arrow[r, "(-)^{hS^1}", "\simeq"'] & \mathrm{Mod}^{\wedge}_{k[\![t]\!]} \\
& \left(\mathrm{Mod}_k^{\mathrm{B}S^1}\right)^{\mathrm{rig}} \arrow[u] &
\end{tikzcd}$$

Construction (Efimov–Scholze)**.** Let k be an $\mathbb{E}_\infty$-ring spectrum. Suppose k is complex orientable, choose complex orientation $t \in \pi_{-2}(k^{hS^1})$.

$$\begin{tikzcd}
\underset{}{\overset{\text{rigid}}{\mathrm{Mot}_k^{\mathrm{loc}}}} \arrow[r, "\mathrm{THH}(-/k)", "\otimes"'] \arrow[dr, dashed, "\mathrm{THH}^{\mathrm{ref}}(-/k)"'] & \overset{\text{not rigid}}{\mathrm{Mod}_k^{BS^1}} \arrow[r, "(-)^{hS^1}", "\simeq"'] & \mathrm{Mod}^{\wedge}_{k[\![t]\!]} \\
& \left(\mathrm{Mod}_k^{BS^1}\right)^{\mathrm{rig}} \arrow[u] \arrow[r, "\simeq"] & \mathrm{Nuc}(k[\![t]\!]) \arrow[u]
\end{tikzcd}$$

Construction (Efimov–Scholze)**.** Let k be an $\mathbb{E}_\infty$-ring spectrum. Suppose k is complex orientable, choose complex orientation $t \in \pi_{-2}(k^{hS^1})$.

$$
\begin{tikzcd}
\overset{\text{rigid}}{\mathrm{Mot}_k^{\mathrm{loc}}} \arrow[r, "\mathrm{THH}(-/k)", "\otimes"'] \arrow[dr, dashed, "\mathrm{THH}^{\mathrm{ref}}(-/k)"'] \arrow[drr, dashed, bend right, "\mathrm{TC}^{-,\mathrm{ref}}(-/k)"'] & \overset{\text{not rigid}}{\mathrm{Mod}_k^{BS^1}} \arrow[r, "(-)^{hS^1}", "\simeq"'] & \mathrm{Mod}^{\wedge}_{k[\![t]\!]} \\
& \left(\mathrm{Mod}_k^{BS^1}\right)^{\mathrm{rig}} \arrow[u] \arrow[r, "\simeq"] & \mathrm{Nuc}(k[\![t]\!]) \arrow[u]
\end{tikzcd}
$$

$\mathrm{THH}^{\mathrm{ref}}$ can be nontrivial modulo p, even for rational inputs.

$\mathrm{THH}^{\mathrm{ref}}$ can be nontrivial modulo p, even for rational inputs. We can describe $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q}) \otimes \mathrm{ku}$

$\mathrm{THH}^{\mathrm{ref}}$ can be nontrivial modulo p, even for rational inputs. We can describe $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q}) \otimes \mathrm{ku}$, or equivalently $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$:

$\mathrm{THH}^{\mathrm{ref}}$ can be nontrivial modulo p, even for rational inputs. We can describe $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q}) \otimes \mathrm{ku}$, or equivalently $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$:

Theorem (Meyer–W. 2024)**.**

$$\pi_* \mathrm{TC}^{-,\mathrm{ref}}(\ \mathrm{ku} \ \ \otimes \mathbb{Q}/ \ \mathrm{ku} \ )$$

$\mathrm{THH}^{\mathrm{ref}}$ can be nontrivial modulo p, even for rational inputs. We can describe $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q}) \otimes \mathrm{ku}$, or equivalently $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$:

Theorem (Meyer–W. 2024)**.**

$$\pi_0 \, \mathrm{TC}^{-,\mathrm{ref}}(\,\mathrm{KU} \;\otimes \mathbb{Q}/\,\mathrm{KU}\,)$$

$\mathrm{THH}^{\mathrm{ref}}$ can be nontrivial modulo p, even for rational inputs. We can describe $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q}) \otimes \mathrm{ku}$, or equivalently $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$:

Theorem (Meyer–W. 2024)**.**

$$\pi_0 \, \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}_p^\wedge \otimes \mathbb{Q}/\mathrm{KU}_p^\wedge)$$

$\mathrm{THH}^{\mathrm{ref}}$ can be nontrivial modulo p, even for rational inputs. We can describe $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q}) \otimes \mathrm{ku}$, or equivalently $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$:

Theorem (Meyer–W. 2024)**.**

$$\pi_0 \, \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}_p^\wedge \otimes \mathbb{Q}/\mathrm{KU}_p^\wedge) \cong \mathcal{O}(Z^\dagger)\,,$$

$\mathrm{THH}^{\mathrm{ref}}$ can be nontrivial modulo p, even for rational inputs. We can describe $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q}) \otimes \mathrm{ku}$, or equivalently $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$:

Theorem (Meyer–W. 2024)**.**

$$\pi_0\, \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}_p^\wedge \otimes \mathbb{Q}/\mathrm{KU}_p^\wedge) \cong \mathcal{O}(Z^\dagger)\,,$$

where

- $\mathcal{O}(-^\dagger)$ denotes a ring of overconvergent functions,

$\mathrm{THH}^{\mathrm{ref}}$ can be nontrivial modulo p, even for rational inputs. We can describe $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q}) \otimes \mathrm{ku}$, or equivalently $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$:

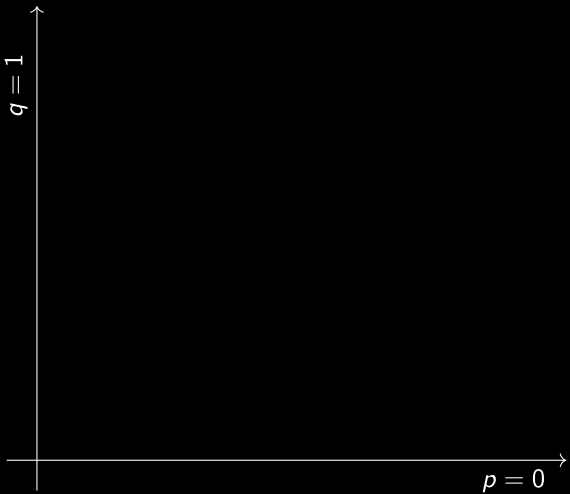
Theorem (Meyer–W. 2024)**.**

$$\pi_0 \, \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}_p^\wedge \otimes \mathbb{Q}/\mathrm{KU}_p^\wedge) \cong \mathcal{O}(Z^\dagger)\,,$$

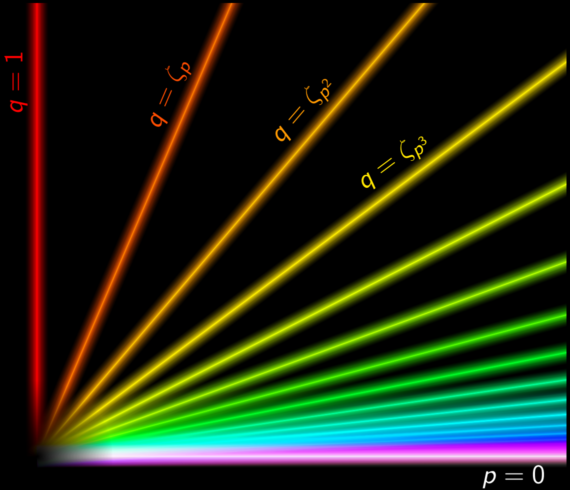
where

- $\mathcal{O}(-^\dagger)$ denotes a ring of overconvergent functions,
- $Z \subseteq \operatorname{Spa} \mathbb{Z}_p[\![q-1]\!]$ denotes the subset

$$\operatorname{Spa}\big(\mathbb{F}_p(\!(q-1)\!), \mathbb{F}_p[\![q-1]\!]\big) \cup \bigcup_{\alpha \geqslant 0} \operatorname{Spa}\big(\mathbb{Q}_p(\zeta_{p^\alpha}), \mathbb{Z}_p[\zeta_{p^\alpha}]\big)\,.$$



spectrum of $\mathbb{Z}_p[\![q-1]\!]$

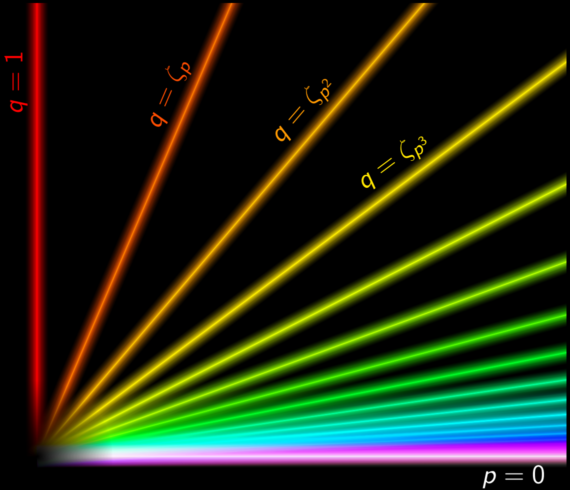


spectrum of $\pi_0\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}_p^\wedge \otimes \mathbb{Q}/\mathrm{KU}_p^\wedge) \cong \mathcal{O}(Z^\dagger)$

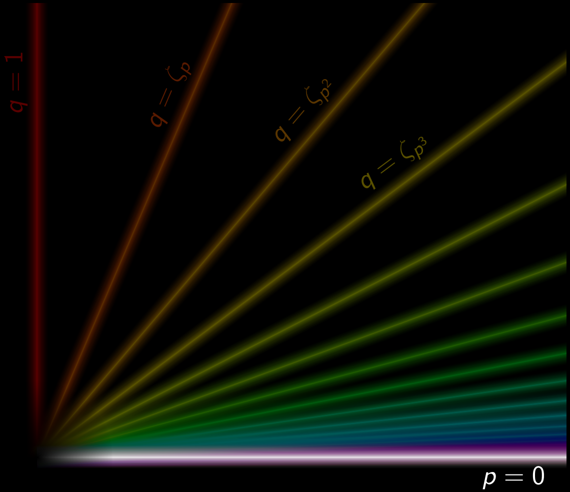
$\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$

MAX PLANCK INSTITUTE
FOR MATHEMATICS

spectrum of $\pi_0 \, \mathrm{TC}^{-}(\mathrm{KU}_p^{\wedge} \otimes \mathbb{Q}/\mathrm{KU}_p^{\wedge}) \cong \mathbb{Q}_p[\![q-1]\!]$



spectrum of $\pi_0\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}_p^\wedge \otimes \mathbb{Q}/\mathrm{KU}_p^\wedge) \cong \mathcal{O}(Z^\dagger)$



spectrum of $\pi_0 \, \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}_p^\wedge \otimes \mathbb{Q}/\mathrm{KU}_p^\wedge) \cong \mathcal{O}(Z^\dagger)$

Analytic Habiro cohomology (Scholze). Coefficients in the *analytic Habiro ring* $\mathcal{H}^{\mathrm{an}}$.

Outlook

MAX PLANCK INSTITUTE
FOR MATHEMATICS

Analytic Habiro cohomology (Scholze)**.** Coefficients in the *analytic Habiro ring* $\mathcal{H}^{\mathrm{an}}$. This is a sheaf on the stack of norms.

Outlook

MAX PLANCK INSTITUTE FOR MATHEMATICS

Analytic Habiro cohomology (Scholze)**.** Coefficients in the *analytic Habiro ring* $\mathcal{H}^{\mathrm{an}}$. This is a $\mathbb{Z}(\!(u)\!)_{\blacksquare}$-algebra, $|u| = \frac{1}{2}$.

Outlook

MAX PLANCK INSTITUTE
FOR MATHEMATICS

Analytic Habiro cohomology (Scholze)**.** Coefficients in the *analytic Habiro ring* $\mathcal{H}^{\mathrm{an}}$. This is a $\mathbb{Z}(\!(u)\!)_{\blacksquare}$-algebra, $|u| = \frac{1}{2}$.

. . . via $\mathbf{THH}^{\mathbf{ref}}$ (work in progress)**.** Crucial observations:

Analytic Habiro cohomology (Scholze)**.** Coefficients in the *analytic Habiro ring* $\mathcal{H}^{\mathrm{an}}$. This is a $\mathbb{Z}(\!(u)\!)_{\blacksquare}$-algebra, $|u| = \frac{1}{2}$.

. . . via THH$^{\mathrm{ref}}$ (work in progress)**.** Crucial observations:

1. (Scholze) The analytic de Rham stack lifts canonically to $\mathbb{S}(\!(u)\!)_{\blacksquare}$.

Analytic Habiro cohomology (Scholze). Coefficients in the *analytic Habiro ring* $\mathcal{H}^{\mathrm{an}}$. This is a $\mathbb{Z}(\!(u)\!)_{\blacksquare}$-algebra, $|u| = \frac{1}{2}$.

. . . via $\mathbf{THH}^{\mathrm{ref}}$ (work in progress). Crucial observations:

1. (Scholze) The analytic de Rham stack lifts canonically to $\mathbb{S}(\!(u)\!)_{\blacksquare}$.
2. (W.) Analytic Habiro ring can be recovered as

$$\mathcal{H}^{\mathrm{an}} \simeq \pi_0 \mathrm{TCn}^{\mathrm{ref}}\left(\mathrm{KU}(\!(u)\!)/\mathrm{KU}[\![u]\!]\right)\left[\tfrac{1}{u}\right].$$

Analytic Habiro cohomology (Scholze). Coefficients in the *analytic Habiro ring* $\mathcal{H}^{\mathrm{an}}$. This is a $\mathbb{Z}(\!(u)\!)_{\blacksquare}$-algebra, $|u| = \frac{1}{2}$.

. . . via $\mathbf{THH}^{\mathbf{ref}}$ (work in progress). Crucial observations:

1. (Scholze) The analytic de Rham stack lifts canonically to $\mathbb{S}(\!(u)\!)_{\blacksquare}$.
2. (W.) Analytic Habiro ring can be recovered as

$$\mathcal{H}^{\mathrm{an}} \simeq \pi_0 \mathrm{TCn}^{\mathrm{ref}}\left(\mathrm{KU}(\!(u)\!)/\mathrm{KU}[\![u]\!]\right)\left[\tfrac{1}{u}\right].$$

$$\left(\qquad \text{Recall: } \mathrm{TCn}(-) := \lim_{m\in\mathbb{N}} \left(\mathrm{THH}(-)^{C_m}\right)^{h(S^1/C_m)}. \qquad\right)$$

Analytic Habiro cohomology (Scholze). Coefficients in the *analytic Habiro ring* $\mathcal{H}^{\mathrm{an}}$. This is a $\mathbb{Z}(\!(u)\!)_{\blacksquare}$-algebra, $|u| = \frac{1}{2}$.

... via THH$^{\mathrm{ref}}$ (work in progress). Crucial observations:

1. (Scholze) The analytic de Rham stack lifts canonically to $\mathbb{S}(\!(u)\!)_{\blacksquare}$.
2. (W.) Analytic Habiro ring can be recovered as

$$\mathcal{H}^{\mathrm{an}} \simeq \pi_0 \mathrm{TCn}^{\mathrm{ref}}\left(\mathrm{KU}(\!(u)\!)/\mathrm{KU}[\![u]\!]\right)\left[\tfrac{1}{u}\right].$$

Question.

Can we compute this for bases other than KU (e.g. elliptic cohomology, MU, $\mathbb{S}$)?

Thank you!